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Motion driven by surface-tension gradients in a tube lining

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A fluid layer that lines the inner surface of a circular tube has motion induced by axial surface-tension gradients. The lubrication equations for the system are analysed and it is found that even for thin layers the motions differ markedly from those in planar layers. The planar case serves as a class of outer solutions. These approximate solutions are modified by a boundary-layer correction where the mean surface tension is important.

1. Introduction

In the human respiratory system, various of the airways are covered by fluid linings. In the upper respiratory system the underlying tissue is ciliated; the beating of the cilia drives the overlying mucous which acts as a barrier to the invasion of foreign particles into the lungs (Carlson, Johnson & Cavert 1965). Such fluid motions have recently been analysed mathematically (Ross 1971). In the lower respiratory system the smallest sacs of the lungs, called alveoli, are likewise covered with a liquid lining. The underlying tissue is thought (Scarpelli 1968) to emit a surface-active agent that acts to reduce drastically the surface tension of the liquid–air interface present. This reduction in surface tension is believed (Scarpelli 1968) to be responsible for the alveoli remaining inflated rather than collapsing under the influence of large capillary forces. The tubes leading to terminal alveoli, called alveolar ducts, are, likewise, thought to be fluid lined but lack the surfactant emitting ability.

One question that has been raised (Clements 1970) concerns the mechanism used by alveoli to dispose of surplus surface-active material (which is continually being produced). There seems to be no accepted explanation in the physiology literature of the process although it seems clear that surface-tension gradients must produce fluid motions whose sense is from the alveolus outward toward the alveolar duct. The present study might be applied to this situation. It should be pointed out that at this time there is no way to either prove or disprove that this mechanism is the *dominant* contributor to surfactant disposal. However, any surface-tension difference along an interface is sufficient to cause motion, so that the kind of motion envisioned should always be present.

There have been previous investigations of motions driven by imposed surface-tension gradients. Yih (1968) analyses, using the lubrication approximation,

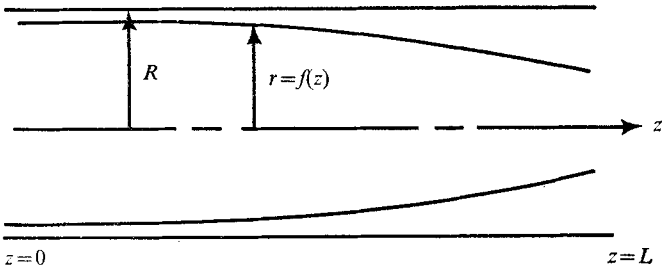

FIGURE 1. The geometry of the system.

a two-dimensional layer on a plane and subject to gravity. Adler & Sowerby (1970) consider a three-dimensional version of Yih's problem. Yih (1969) further considers boundary layers formed by flows driven by surface-tension gradients near vertical walls.

The present study considers the steady axisymmetric motion in a Newtonian fluid lining in a circular tube in the absence of gravity. The motion is driven by axial surface-tension gradients generated by axial gradients of the concentration of surface-active material. The full governing equations are analysed using lubrication theory, under which the equations look similar to those for creeping motion. The lubrication equations are first solved in the limit of vanishing thickness-to-radius ratio δ (planar limit). This limit regains Yih's (1968) equations (properly interpreted) for the case of zero gravity. It is seen that the planar layer *thins* in the direction of the motion (i.e. in the direction of the surface-tension gradient). Solution curves are presented. The dynamics of the fluid lining of a tube, in general, significantly differ from those of a layer on a plane even for thin layers. The differences originate from the curvature κ of the interface around the tube. If σ is the absolute surface tension at a point on the interface, the curvature κ gives rise to a pressure field $\kappa\sigma$ (approximately) which has a non-zero axial gradient. This pressure gradient modifies the flow from that in a planar layer even for small δ by creating a boundary layer at one end of the tube and modifying the flow rates throughout. Thus, in a fluid lining of a tube, the motion depends not only on the axial *surface-tension difference* but also the *absolute surface tension*. The limit $\delta \to 0$ is seen to be a singular one. Both an asymptotic analysis and numerical solutions are presented.

2. Formulation

Let us consider a Newtonian fluid of constant density ρ_0 that forms a lining of a circular tube of radius R and length L as shown in figure 1. We shall seek to describe the motion induced in this lining when an insoluble surfactant of variable surface concentration Γ is present. We shall use cylindrical co-ordinates to describe the motion. The z axis lies along the centre-line of the tube and the concentration $\Gamma = \Gamma_1$ is imposed at the end $z = 0$ while $\Gamma = 0$, say, is imposed at the end $z = L$. We shall seek motions that are steady, axially symmetric and for which gravity can be neglected. The velocity components in the (r, θ, z)

directions are given by $(u, 0, w)$ and the surface of the liquid lining is described by $r = f(z) \leqslant R$. The motion is governed by the Navier–Stokes and continuity equations and appropriate boundary conditions.

The Navier–Stokes and continuity equations reduce to the following:

$$uu_r + wu_z = -(p/\rho_0)_r + \nu(u_{rr} + r^{-1}u_r - r^{-2}u), \tag{2.1a}$$

$$uw_r + ww_z = -(p/\rho_0)_z + \nu(w_{rr} + r^{-1}w_r), \tag{2.1b}$$

$$u_r + r^{-1}u + w_z = 0. \tag{2.1c}$$

The boundary conditions on the solid wall are

$$u = w = 0 \quad \text{on} \quad r = R, \tag{2.1d}$$

while on the interface $r = f(z)$, the kinematic boundary condition reduces to

$$u = wf_z. \tag{2.1e}$$

In addition, the interface is endowed with a surface tension σ which is allowed to vary with surface concentration Γ of surfactant as follows:

$$\sigma = \sigma_2 - \alpha\Gamma \quad (\alpha > 0). \tag{2.1f}$$

The jump in normal stress across the interface is balanced by the surface tension times the curvature κ while the jump in the shear stress is balanced by the surface gradient of surface tension and surface viscous forces. One model of these conditions can be written (Aris 1962, p. 242) as follows:

$$\mathbf{T}.\mathbf{n} + \nabla_s\sigma + \kappa\sigma\mathbf{n} + \mu_1\nabla_s^2\mathbf{u}_s + \mu_2\nabla_s(\nabla_s.\mathbf{u}_s) = \mathbf{0} \quad \text{on} \quad r = f(z). \tag{2.1g}$$

Here $\mathbf{T}$ is the stress tensor of the bulk fluid, $\mathbf{n}$ is a unit inward normal vector, ∇_s is the gradient along the interface and $\mathbf{u}_s$ is the surface velocity vector. μ_1 and μ_2 are the shear and dilatational surface viscosities respectively. In addition, since the surface-active material is confined to the surface, a balance equation for it takes the form

$$\nabla_s.(\Gamma\mathbf{u}_s) = \mathscr{D}_s\nabla_s^2\Gamma, \tag{2.1h}$$

where $\mathscr{D}_s$ is the surface diffusivity coefficient. Sealy (1972) has considered the effects of solubility and concluded through order-of-magnitude estimates that these effects are small in the present problem.

We shall seek a solution of system (2.1) using the lubrication approximation, which results in equations that are inertia-free. A *sufficient* condition which gives these equations is obtained by a formal expansion in, say, λ, some measure of the slope of the interface. The approximation results from the leading term of the following formal expansions:

$$\left.\begin{aligned} \sigma = \sigma^{(0)} + O(\lambda), \quad \Gamma = \Gamma^{(0)} + O(\lambda), \quad p = p^{(0)} + O(\lambda), \\ w = \lambda w^{(1)} + O(\lambda^2), \quad u = \lambda^2 u^{(2)} + O(\lambda^3), \quad f = f^{(0)} + O(\lambda). \end{aligned}\right\} \tag{2.2}$$

The resulting differential system with superscripts dropped has the form:

$$\left.\begin{aligned} p_r &= 0 \\ \mu(w_{rr} + r^{-1}w_r) - p_z &= 0 \\ u_r + r^{-1}u + w_z &= 0 \end{aligned}\right\} \quad \text{in } f(z) \leqslant r \leqslant R, \tag{2.3a, b, c}$$

$$\left.\begin{array}{l} -\mu w_r = \dfrac{d\sigma}{dz}, \quad u = w\dfrac{df}{dz}, \quad p = -\dfrac{\sigma}{f} \\ d(w\Gamma)/dz = \mathscr{D}_s d^2\Gamma/dz^2, \quad \sigma = \sigma_2 - \alpha\Gamma \end{array}\right\} \quad \text{on } r = f(z), \qquad (2.3d\text{–}h)$$

$$u = w = 0 \quad \text{on} \quad r = R. \qquad (2.3i)$$

The precise derivation of these equations involves some elementary differential geometry and is given in detail in Sealy (1972). Γ and σ are defined only along the interface and d/dz of these quantities denotes the directional derivative in the direction of the local tangent vector, which, within the lubrication approximation, is the ordinary derivative of these quantities treated as functions of arc length. When d/dz is applied to a function such as f which depends on z only, the directional derivative can be computed as follows:

$$\frac{df}{dz} = \left(\frac{\partial}{\partial z} + \frac{df}{dz}\frac{\partial}{\partial r}\right) f = \frac{\partial f}{\partial z}$$

and so degenerates to the usual ordinary derivative operator.

Within this approximation, the normal stress boundary condition reduces to the classical one, given in (2.3f). The tangential stress boundary condition reduces to a balance between the shear stress in the bulk at the surface and the surface-tension gradient (2.3d). *Within the lubrication approximation, contributions* to this balance *due to surface viscosities are small.*

3. Solutions

Equations (2.3a, f) show that the curvature in the θ-direction gives rise to a pressure field which is uniform in r but which varies in the axial direction:

$$p(z) = -\sigma(z)/f(z). \qquad (3.1a)$$

The general solution of the z-momentum equation (2.3b) that satisfies conditions (2.3i) and (2.3d) is

$$\mu w(r,z) = -\frac{1}{4}\frac{dp}{dz}\left\{R^2 - r^2 + 2f^2 \ln\frac{r}{R}\right\} - f\frac{d\sigma}{dz}\ln\frac{r}{R}. \qquad (3.1b)$$

We can replace the continuity condition (2.3c), using $u = 0$ on $r = R$, by the equivalent integral condition

$$\int_{f(z)}^{R} rw(r,z)\,dr = Q_1,$$

which, if (3.1b) is used, reduces to

$$\mu Q_1 = -\frac{1}{4}\frac{dp}{dz}\left\{\tfrac{1}{4}(R^2-f^2)(R^2-3f^2) - f^4 \ln\frac{f}{R}\right\} + f\frac{d\sigma}{dz}\left\{\tfrac{1}{4}(R^2-f^2) + \tfrac{1}{2}f^2 \ln\frac{f}{R}\right\}. \qquad (3.1c)$$

The surface diffusion equation (2.3g) can be integrated to yield

$$\Gamma w - \mathscr{D}_s d\Gamma/dz = q_\Gamma,$$

where q_Γ is the surface flux of the surfactant. We can eliminate Γ through (2.3h) and obtain

$$(\sigma - \sigma_2)\,w - \mathscr{D}_s d\sigma/dz = q_1,$$

where $q_1 = -\alpha q_\Gamma$ and we note that q_1 and q_Γ have opposite signs. If we replace w by its value given in (3.1b), we obtain

$$\mu q_1 = -\mu\mathscr{D}_s\frac{d\sigma}{dz}+(\sigma-\sigma_2)\left\{-\frac{1}{4}\frac{dp}{dz}\left[R^2-f^2+2f^2\ln\frac{f}{R}\right]-f\frac{d\sigma}{dz}\ln\frac{f}{R}\right\}. \quad (3.1d)$$

The pair of equations (3.1c, d) is a linked set of ordinary differential equations for the functions σ and f when the pressure gradient dp/dz is replaced by its value obtained from (3.1a). These have the form

$$-\mu Q_1 = -\frac{1}{4f}\left\{\tfrac{1}{4}(R^4-f^4)+f^4\ln\frac{f}{R}\right\}\frac{d\sigma}{dz}$$
$$+\frac{\sigma}{4f^2}\left\{\tfrac{1}{4}(R^2-f^2)(R^2-3f^2)-f^4\ln\frac{f}{R}\right\}\frac{df}{dz}, \quad (3.2a)$$

$$-\mu q_1 = \left\{\mu\mathscr{D}_s-\frac{(\sigma-\sigma_2)}{4f}\left[R^2-f^2-2f^2\ln\frac{f}{R}\right]\right\}\frac{d\sigma}{dz}$$
$$+\tfrac{1}{4}\sigma(\sigma-\sigma_2)\frac{1}{f^2}\left\{R^2-f^2+2f^2\ln\frac{f}{R}\right\}\frac{df}{dz}. \quad (3.2b)$$

The velocity w takes the form

$$\mu w = \frac{1}{4f}\left\{R^2-r^2-2f^2\ln\frac{r}{R}\right\}\frac{d\sigma}{dz}-\frac{\sigma}{4f^2}\left\{R^2-r^2+2f^2\ln\frac{r}{R}\right\}\frac{df}{dz}. \quad (3.2c)$$

4. Scaling

Before turning to the solution, let us scale the equations governing the solutions in such a way that it will be convenient later to look at the limiting case where the lining is thin compared with the tube radius.

Let us define h_1,

$$h_1 = R-f(0), \quad (4.1a)$$

as the lining thickness at $z = 0$. We shall presume, and verify numerically later, that $R-f(z)$ is of order h_1 for all z. The lining thickness $h'(z')$ can then be defined in terms of $z' = z/L$:

$$h'(z') = (R-f(z))/(R-f(0)), \quad (4.1b)$$

so that $h'(0) = 1$. The thickness ratio δ is defined by

$$\delta = h_1/R \quad (4.1c)$$

and the lining is termed *thin* if

$$\delta \ll 1.$$

We can now define a non-dimensional radial co-ordinate x' measured in units of h_1 as follows:

$$r = R\{1-\delta h'(z')x'\}, \quad (4.1d)$$

so that

$$x' = \begin{cases} 0 & \text{if } r = R, \\ 1 & \text{if } r = f(z). \end{cases} \quad (4.1e)$$

The general problem posed in § 3 has two surface-tension contributions. The surface-tension *difference* from one end of the tube to the other drives the motion but in addition, the curvature in the θ-direction contributes an *axial pressure gradient* depending on the *absolute surface tension* that modifies the flow. As a result, separate scales are needed for absolute surface tension and surface-tension differences. If σ_1 and σ_2 are the values of the surface tensions at $z' = 0$ and $z' = 1$ respectively, a non-dimensional surface-tension *difference* σ' can be defined as follows:

$$\sigma = \sigma_2 + (\sigma_2 - \sigma_1)\,\sigma'.$$

The first term (one might use $\frac{1}{2}(\sigma_1+\sigma_2)$ here instead of σ_2) is a measure of the absolute surface tension while the following term is a measure of the surface-tension difference. It follows that on the interface

$$\sigma(z) = (\sigma_2 - \sigma_1)\,\{S + \sigma'(z')\}, \tag{4.1f}$$

$$S = \sigma_2/(\sigma_2 - \sigma_1) \tag{4.1g}$$

and

$$\sigma'(0) = -1, \quad \sigma'(1) = 0. \tag{4.1h,i}$$

If definitions (4.1) are substituted into (3.2a) and (3.2b), a pair of differential equations defining σ' and h' is obtained. If (3.2a) is divided by δ^2 and (3.2b) is divided by δ, the following system results:

$$\left.\begin{array}{c}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} d\sigma'/dz' \\ dh'/dz' \end{pmatrix} = \begin{pmatrix} Q \\ q \end{pmatrix} \quad \text{on} \quad 0 \leqslant z' \leqslant 1, \\ \sigma'(0) = -1, \quad \sigma'(1) = 0, \quad h'(0) = 1, \end{array}\right\} \tag{4.2}$$

where

$$Q = L\mu Q_1/(\sigma_2 - \sigma_1)\,h_1^2 R, \quad q = L\mu q_1/(\sigma_2 - \sigma_1)^2 h_1, \quad \eta = \mu\mathscr{D}_s/(\sigma_2 - \sigma_1)\,h_1,$$

$$A_{11} = \tfrac{1}{4}\delta^{-2}(1-\delta h')^{-1}\{\tfrac{1}{4}[1-(1-\delta h')^4] + (1-\delta h')^4 \ln(1-\delta h')\},$$

$$A_{12} = \tfrac{1}{4}\delta^{-1}(S+\sigma')\,(1-\delta h')^{-2}\{\tfrac{1}{4}[1-(1-\delta h')^2]\,[1-3(1-\delta h')^2] - (1-\delta h')^4 \ln(1-\delta h')\},$$

$$A_{21} = -\eta + \tfrac{1}{4}\delta^{-1}\sigma'(1-\delta h')^{-1}\{1-(1-\delta h')^2 - 2(1-\delta h')^2 \ln(1-\delta h')\},$$

$$A_{22} = \tfrac{1}{4}\sigma'(S+\sigma')\,(1-\delta h')^{-2}\{1-(1-\delta h')^2 + 2(1-\delta h')^2 \ln(1-\delta h')\}.$$

5. Estimates of the values of the parameters in an alveolar situation

In an alveolus, the surfactant material lowers the local surface tension from say 75 dyne/cm to, perhaps, a value of 25 dyne/cm or even lower. Take, then,

$$\sigma_1 = 25\,\text{dynes/cm}, \quad \sigma_2 = 75\,\text{dynes/cm},$$

so that

$$S = 1{\cdot}5.$$

An alveolus has a diameter of perhaps 200 μm so let us take the radius of the alveolar duct as 100 μm and the fluid layer of depth 10 μm. This last number is totally unverified and hence should be regarded as only a rough order of magnitude. Hence,

$$\delta = 0{\cdot}10. \tag{5.1a}$$

The surface diffusion constant $\mathscr{D}_s$ is only roughly known. Say,

$$\mathscr{D}_s = 10^{-4}\,\mathrm{cm}^2/\mathrm{s}$$

as estimated for similar materials by Sakata & Berg (1969). If μ is taken to be the viscosity of water, then

$$\eta = 2\times 10^{-5}. \tag{5.1b}$$

As will be seen, the solutions depend only weakly on η for η values this small.

6. Thin layers

If it is remembered that q is the 'surface-tension flux' and is of opposite sign from the surface flux of surfactant, then it should be expected that $q < 0$.

Let us introduce the ratio R_Q:

$$R_Q = -q/Q. \tag{6.1a}$$

If the A_{ij} of system (4.2) are expanded in powers of δ, the leading terms can be written using (6.1*a*) as follows:

$$\left.\begin{aligned} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22}\end{pmatrix}\begin{pmatrix} d\sigma'/dz' \\ dh'/dz'\end{pmatrix} &= Q\begin{pmatrix} 1 \\ -R_Q\end{pmatrix}, \\ \sigma'(0) = -1, \quad \sigma'(1) = 0, \quad & h'(0) = 1,\end{aligned}\right\} \tag{6.1b}$$

where

$$B_{11} \sim \tfrac{1}{2}h'^2 - \tfrac{1}{3}h'^3\delta + \tfrac{1}{8}h'^4\delta^2,$$

$$B_{12} \sim \tfrac{1}{3}(S+\sigma')\,h'^3\delta^2,$$

$$B_{21} \sim -\eta + \sigma'(h' + \tfrac{1}{6}h'^2\delta + \tfrac{5}{24}h'^3\delta^2),$$

$$B_{22} \sim \tfrac{1}{2}(S+\sigma')\,\sigma' h'^2\delta^2$$

and we have omitted terms $O(\delta^3)$ as $\delta \to 0$. These are the approximate equations that govern the motions in thin layers.

Before proceeding with the solution of system (6.1), let us examine some properties of the system.

The matrix $\mathbf{B}$ (as is the matrix $\mathbf{A}$ of system (4.2)) is explicitly independent of z'. Hence, the phase plane equation obtained by inverting $\mathbf{B}$ (if this is possible) can be used. This has the form

$$dh'/d\sigma' = -(B_{21} + R_Q B_{11})/(B_{22} + R_Q B_{12}). \tag{6.2a}$$

The corresponding boundary conditions are as follows:

$$h' = \begin{cases} 1 & \text{at} \quad \sigma' = -1, \quad (6.2b) \\ h'(1) & \text{at} \quad \sigma' = 0. \quad (6.2c)\end{cases}$$

7. Planar limit

If it is assumed that $dh'/dz' = O(1)$ as $\delta \to 0$, the *formal* limit $\delta \to 0$ of (6.1) gives the system

$$\tfrac{1}{2}h'^2\, d\sigma'/dz' = Q, \tag{7.1a}$$

$$(\sigma' h' - \eta)\, d\sigma'/dz' = -QR_Q, \tag{7.1b}$$

which agrees with those equations derived by Yih (1968) for two-dimensional planar motions driven by surface-tension gradients in the limit of zero gravity. Yih discusses the possibility of (7.1*b*) being singular at the point where $\sigma' h' - \eta = 0$ since he mistakenly treats σ' as the absolute surface tension and not the surface-tension difference. As we have seen, with the scaling given in § 4, $\sigma' \leqslant 0$, so that no singularity of this kind can occur. This was first recognized by Adler & Sowerby (1970).

The limit $\delta \to 0$ with Q, R_Q, η and S fixed is a singular limit of system (6.1) in the sense that the matrix **B** becomes singular (non-invertible) there. dh'/dz' does not appear in the governing system (7.1) and the absolute surface tension no longer has an effect. (The equations are independent of S.)

If we take the ratio of (7.1*a*) and (7.1*b*), we obtain a compatibility (algebraic) equation for σ' and h':

$$2(\sigma' h' - \eta)/h'^2 = -R_Q. \tag{7.1c}$$

The ratio R_Q is determined from the condition that $h' = 1$ when $\sigma' = -1$:

$$R_Q = 2(1+\eta). \tag{7.1d}$$

Hence, the final value $h'(1)$ of the height can be obtained directly from (7.1*c*), (7.1*d*) and $\sigma'(1) = 0$:

$$h'(1) = [\eta/(1+\eta)]^{\frac{1}{2}}. \tag{7.2a}$$

The final thickness is much smaller than the initial thickness when, as is usual, $n \ll 1$.

The solution for the algebraic equation (7.1*c*) is as follows:

$$h' = R_Q^{-1}\{|\sigma'| + [\sigma'^2 + 2\eta R_Q]^{\frac{1}{2}}\}. \tag{7.2b}$$

The limit $\eta \to 0$ decreases the order of the surface diffusion equation from two to one, so that a diffusion boundary layer can be expected. For $\eta \ll 1$, there is such a layer at $z' = 1$, where $\sigma' = 0$. It has σ'-thickness $O(\eta^{\frac{1}{2}})$ and its presence changes $h'(1)$ from zero to $[\eta/(\eta+1)]^{\frac{1}{2}}$ as given by (7.2*a*). Figure 2 illustrates the variation of h' with σ' for various values of η.

The z' dependences of h' and σ' can be obtained by substituting (7.2*b*) into (7.2*a*) and enforcing $\sigma'(0) = -1$. The result is as follows:

$$3\eta R_Q(|\sigma'|-1) + (|\sigma'|^3 - 1) + (|\sigma'|^2 + 2\eta R_Q)^{\frac{3}{2}} - (1 + 2\eta R_Q)^{\frac{3}{2}} = -3QR_Q^2\, z'. \tag{7.2c}$$

The value of Q (and hence through R_Q, the value of q) is obtained by imposing the condition $\sigma'(1) = 0$.

It is easy to show that for small η

$$-q = \tfrac{1}{3}\{1 + 5\eta + 2\eta^2 + O(\eta^3)\},$$

$$Q = \tfrac{1}{6}\{1 + 4\eta - 2\eta^2 + O(\eta^3)\}.$$

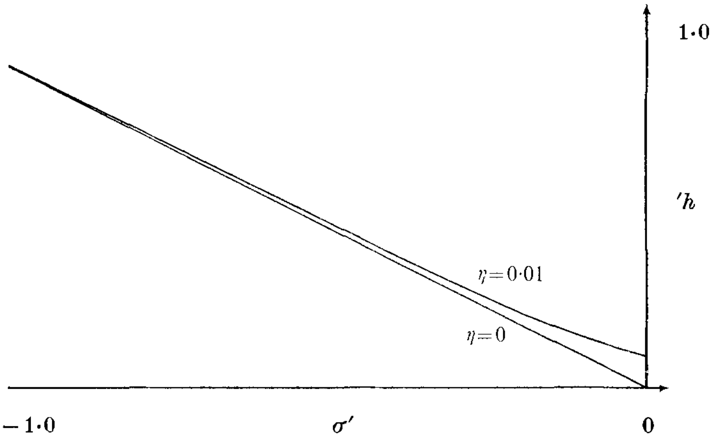

FIGURE 2. The curves h' *vs.* σ' in the planar limit for $\eta = 0$ and $\eta = 0{\cdot}01$.

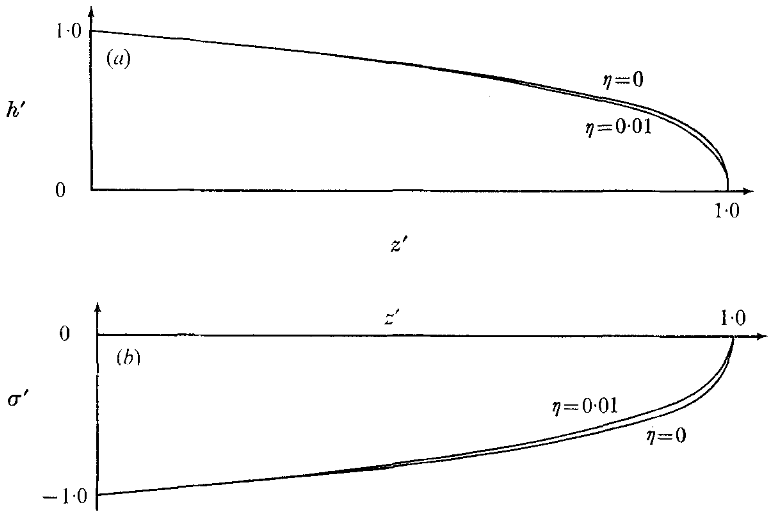

FIGURE 3. The curves (*a*) h' *vs.* z' and (*b*) σ' *vs.* z' in the planar limit for $\eta = 0$ and $\eta = 0{\cdot}01$.

Figures 3(*a*) and (*b*) give the z' variations of h' and σ' respectively. Near $z' = 1$, h' exhibits a surface diffusion boundary layer of thickness $O(\eta^{\frac{1}{2}})$ which results in the finite downstream thickness given in (7.2*a*). The curve of σ' *vs.* z' exhibits no such structure. These figures show numerical solutions for $\eta = 0$ and $\eta = 0{\cdot}01$, the latter of which is an exceedingly large value of η, much larger than could be realized physiologically. We use it here since the η dependence of h' and σ' is so weak that realistic values would not give discernible differences in the figures. The Runge–Kutta–Gill procedure was applied to the equation with $-1 \leqslant \sigma' \leqslant 0$ as independent variable and z' as dependent variable. This guaranteed that the gradients were small enough for accurate solution. These results agreed well with the analytical solution and thus served as a test on the procedure that is used for the non-planar equations.

The result that the layer thins in the direction of motion is easily understood. When $\delta = 0$, lubrication theory gives that the flow is *locally* plane and parallel; in fact, it is the plane Couette flow driven by the surface shear stress imposed by the surface-tension gradient:

$$w(x', z') = (d\sigma'/dz')\, x'.$$

On the free surface when η is neglected, we have that

$$w\sigma' = q.$$

σ' and the constant q are negative. Since σ' decreases in magnitude as z' increases, w increases. The surface fluid speeds up as z' increases. Since the bulk flow is plane Couette flow, the average axial velocity increases with z'. By continuity, the thickness must then decrease with z'.

8. Non-planar flows

For small but non-zero δ, the matrix **B** in system (6.1) is generally non-singular, so that the inverted system can be treated.

$$\frac{d}{dz'}\begin{pmatrix}\sigma' \\ h'\end{pmatrix} = \frac{Q}{\Delta}\begin{pmatrix} B_{22} + R_Q B_{12} \\ -B_{21} - R_Q B_{11}\end{pmatrix}, \tag{8.1}$$

where $$\Delta = B_{11}B_{22} - B_{12}B_{21}.$$

The right-hand side of system (8.1) is explicitly independent of z', so that we can consider h' as a function of σ' only, by dividing one of equations (8.1) by the other. This is the system (6.2). The leading term as $\delta \to 0$ can be written explicitly as follows:

$$\delta^2 \frac{dh'}{d\sigma'} = -\frac{1}{(S+\sigma')h'} \frac{-\eta + h'(\sigma' + \frac{1}{2}R_Q h')}{\frac{1}{2}\sigma' + \frac{1}{3}R_Q h'}. \tag{8.2}$$

The initial slope (at $z' = 0$, $h' = 1$, $\sigma' = -1$) of h' as a function of σ' is easily written down as a function of R_Q and is shown in figure 4. The slope is infinite at $R_Q = R_\infty = \frac{3}{2}$ as $\delta \to 0$ and is zero at $R_Q = R_0 = 2(1+\eta)$ as $\delta \to 0$. The value $R_Q = 2(1+\eta)$ is seen to be the characteristic value of R_Q in the *planar limit* as given in (7.1*d*).

The singular nature of the problem as $\delta \to 0$ is apparent from (8.2). Approximate solutions valid asymptotically as $\delta \to 0$ can be constructed as follows.

Consider an *outer region* where $dh'/d\sigma' = O(1)$ as $\delta \to 0$. Then, in order to have $\delta \to 0$, we must take

$$-\eta + h'(\sigma' + \tfrac{1}{2}R_Q h') \equiv 0.$$

As a result $$h' = R_Q^{-1}\{|\sigma'| + (\sigma'^2 + 2\eta R_Q)^{\frac{1}{2}}\}. \tag{8.3}$$

This is precisely the planar-limit relation (7.2*b*). However, the relation (8.3) is *not* a valid approximation to the solution of (8.2) for all σ'. This is easily seen by comparing the initial slopes $[dh'/d\sigma']_{(-1,1)}$. Figure 4 shows initial slopes of the solutions of (8.2) as $O(\delta^{-2})$ as $\delta \to 0$ while those computed from (8.3) are $O(1)$. There is a boundary layer near $\sigma' = -1$ (i.e. $z' = 0$) in which $dh'/d\sigma'$ is large.

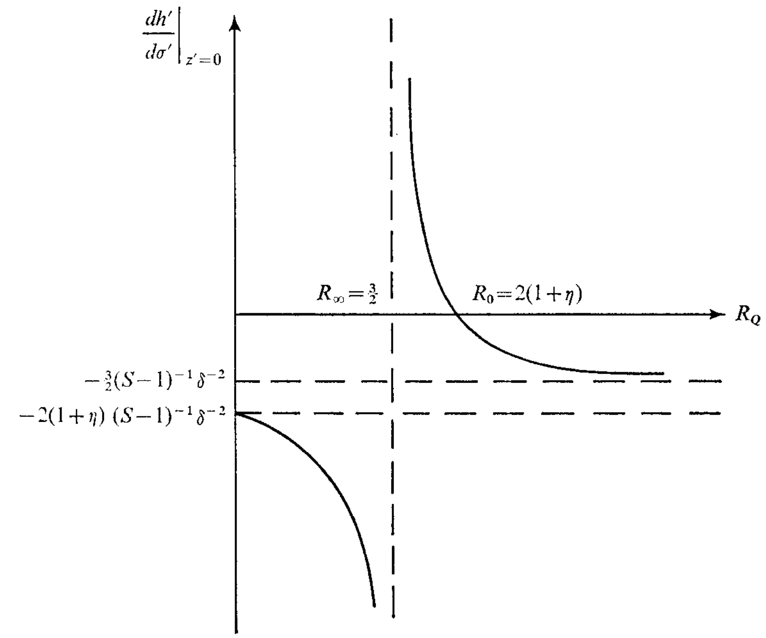

FIGURE 4. The initial value of the slope $dh'/d\sigma'$ *vs.* R_Q for the thin-layer equations keeping the dominant terms in δ. At $R_Q = R_\infty = \frac{3}{2}$, the initial slope is unbounded; at
$$R_Q = R_0 = 2(1+\eta),$$
the initial slope is zero.

Hence, we cannot impose the 'inner' boundary condition $h' = 1$ at $\sigma' = -1$ to obtain the value of R_Q.

In order to examine the *inner region* near $\sigma' = -1$, the stretched co-ordinate Σ' is introduced:
$$\Sigma' = (1+\sigma')/(S-1)\,\delta^2. \tag{8.4}$$

An inner approximation to the solutions of (8.2) is governed to leading order by the following equation:
$$\frac{dh'}{d\Sigma'} = -\frac{1}{h'^2}\frac{-\eta + h'(-1+\frac{1}{2}R_Q h')}{-\frac{1}{2}+\frac{1}{3}R_Q h'}.$$

The solutions are easily obtained for all η but when $\eta \neq 0$ they are cumbersome and differ only slightly from the $\eta = 0$ solution. (The η boundary layer is near $\sigma' = 0$.) When $\eta = 0$, the solution has the form
$$\left|\frac{h' - 2/R_Q}{1-2/R_Q}\right|^{2/R_Q} \exp\left[(h'^2-1) + R_Q^{-1}(h'-1)\right] = \exp\left[-\tfrac{4}{3}\Sigma'\right] \tag{8.5}$$

and satisfies $h' = 1$ at $\Sigma' = 0$.

The matching condition is given by
$$h' \to 2/R_Q \quad \text{as} \quad \Sigma' \to \infty.$$

The inner solution (8.5) *automatically* matches the outer solution (8.3) as $\sigma' \to -1$. A composite solution is easily constructed by adding the inner and outer

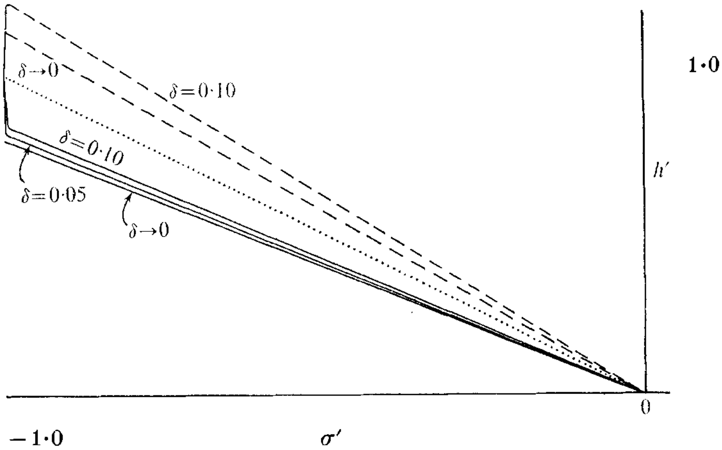

FIGURE 5. The curves of h' *vs.* σ' for the thin-layer equations, $\eta = 0$, $S = 1{\cdot}5$. ——, $R_Q = 2{\cdot}5$; – – –, $R_Q = 1{\cdot}75$. ·····, $R_Q = 2{\cdot}0$. The boundary-layer correction near $\sigma' = -1$ has been left off the asymptotic result $\delta \to 0$ for clarity.

solutions and subtracting the common part $2/R_Q$. The automatic matching holds not for a single value of R_Q but for the whole range of R_Q, $R_Q > \frac{3}{2}$.

The system (6.2) is independent of Q; it depends only on the ratio R_Q. This system has been numerically integrated for $\eta = 0$ (small non-zero η introduces a thin-boundary-layer correction near $\sigma' = 0$) and $S = 1{\cdot}5$. The results are shown in figure 5. The topmost (solid) curve is for $R_Q = 2{\cdot}5$ and $\delta = 0{\cdot}10$, the middle solid curve is for $\delta = 0{\cdot}05$ while the bottommost solid curve is the asymptotic approximation valid for $R_Q = 2{\cdot}5$ and $\delta \to 0$ in the outer region. This is, of course, corrected near $\sigma' = -1$ by a thin boundary layer which is left off the graph for purposes of clarity. The agreement between the analytical asymptote and the numerical solution is excellent. A similar set of curves for the case $R_Q = 1{\cdot}75$ and $\delta = 0{\cdot}10$ is shown dashed in figure 5. It can be seen that a whole family of planar solutions (8.3) (parameterized by R_Q) are good approximations to non-planar ones away from $\sigma' = -1$ and each of these is corrected near $\sigma' = -1$ by a boundary layer of thickness $O((S-1)\,\delta^2)$. The family of solutions varies continuously with R_Q from large R_Q, down to $R_Q = 2$ (where a boundary layer is unnecessary), down towards $R_Q = R_\infty = \frac{3}{2}$. At R_∞, the initial slope of h' is ∞ and a boundary layer cannot be accommodated. For $R_Q < R_\infty$, the initial slope is negative while a boundary layer extending upward would be required. No solution for $R_Q < \frac{3}{2}$ could be obtained either analytically or numerically.

Physically, the outer region is one where only surface-tension *differences* are important. (The solutions are well approximated by a family of planar solutions.) The *mean* surface tension is important only in (inner) regions where non-planar effects enter, the boundary layer near $\sigma' = -1$. The boundary-layer thickness at this upstream end is $O((S-1)\,\delta^2)$ and $S-1$ is a non-dimensional form of the local mean surface tension.

We now turn to the system (8.1) for the z' dependences of σ' and h'. The value of Q will be determined in terms of R_Q.

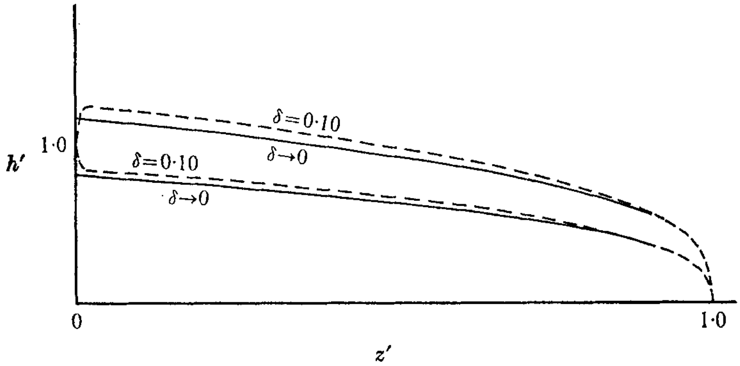

FIGURE 6. The curves of h' *vs.* z' for the thin-layer equations, $\eta = 0$, $S = 1{\cdot}5$, $\delta = 0{\cdot}10$. ——, $R_Q = 2{\cdot}5$; – – –, $R_Q = 1{\cdot}75$. The boundary layer near $z' = 0$ has been left off the asymptotic result $\delta \to 0$ for clarity.

The outer solution valid away from $z' = 0$ is given by the planar equations and for $\eta = 0$ is

$$h'(z') = (12Q/R_Q)^{\frac{1}{3}}(1-z')^{\frac{1}{3}}, \tag{8.6}$$

where h' satisfies the appropriate condition $h'(1) = 0$ in the outer region. Again, if η is small but non-zero, the solution (8.6) is modified by an $O(\eta^{\frac{1}{2}})$ boundary layer near $z' = 1$. We have chosen to discuss the $\eta = 0$ case merely for clarity. The outer solution (8.6) should match an inner solution valid near $z' = 0$; the matching value is

$$h'(0) = (12Q/R_Q)^{\frac{1}{3}}. \tag{8.7}$$

In the inner region, the appropriate stretched co-ordinate is ζ',

$$\zeta' = z'/(S-1)\,\delta^2. \tag{8.8}$$

This is valid near $\sigma' = -1$, so that the appropriate boundary-layer equation is

$$\frac{dh'}{d\zeta'} = -6QR_Q\frac{h'-2/R_Q}{h'^3}. \tag{8.9}$$

The solution of (8.9) that satisfies $h' = 1$ at $\zeta' = 0$ is

$$\left|\frac{h'-2/R_Q}{1-2/R_Q}\right|^{8/R_Q^3}\exp\left[\tfrac{1}{3}(h'^3-1)+R_Q^{-1}(h'^2-1)+4R_Q^{-2}(h'-1)\right] = \exp\left[-6QR_Q\zeta'\right]. \tag{8.10}$$

In the matching region,

$$h' \to 2/R_Q \quad \text{as} \quad \zeta' \to \infty. \tag{8.11}$$

The matching principle states that the values given in (8.7) and (8.11) should agree. Hence, Q is determined:

$$Q = 2/3R_Q^2. \tag{8.12}$$

With the scalings (8.4) and (8.8) it is easy to verify that the graph of σ' *vs.* z' does not display a boundary-layer character; the variations near $z' = 0$ are gentle.

Equations (6.1) were numerically integrated for $\eta = 0$, $\delta = 0{\cdot}10$ and $S = 1{\cdot}5$ for two cases: $R_Q = 2{\cdot}5$ and $R_Q = 1{\cdot}75$. Figure 6 shows h' *vs.* z' compared with

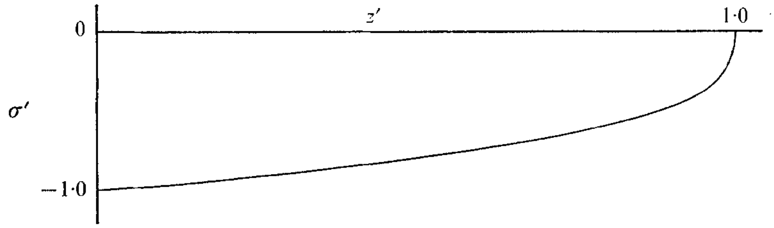

FIGURE 7. The curves of σ' *vs.* z' for the thin-layer equations, $\eta = 0$, $S = 1{\cdot}5$, $\delta = 0{\cdot}10$, for both $R_Q = 2{\cdot}5$ and $R_Q = 1{\cdot}75$.

the outer (planar) approximation and again the boundary-layer correction has been omitted for clarity. Figure 7 shows the corresponding σ' *vs.* z' curves with its lack of boundary-layer structure. The two curves appear here as a single curve since the values are so close.

The full system of equations (4.2) with no small-δ approximation was integrated numerically for several cases in order to test the reliability of the thin-layer approximation. Since system (6.1) was obtained from system (4.2) by neglecting terms $O(\delta^3)$, as expected, the numerical solutions for the two systems differed in the third decimal place for the case tested, i.e. $\delta = 0{\cdot}10$.

9. Conclusions

Motions in fluid layers lining circular cylindrical tubes have been examined. These flows are assumed axisymmetric and steady and are driven by axial surface-tension gradients produced by the presence of a distribution of insoluble surfactant. The lubrication approximation (inertia-free, locally parallel flow) has been envoked and for thin layers, the further approximation $\delta \ll 1$ has been used.

The formal limit $\delta \to 0$ reproduces the planar case treated by Yih (1968). This solution as expected displays no dependence on the local mean surface tension but depends only on the local surface-tension gradient. The velocity field is locally plane Couette flow and the layer thins in the direction of motion.

When $\delta \ll 1$ but $\delta \neq 0$, the flow is markedly different from the planar case. The limit $\delta \to 0$ is a singular one. For small δ, the solutions display a boundary-layer character near the low-surface-tension end of the tube. Outside the boundary layer, the solutions are well approximated by $\delta = 0$ solutions. However, since the outer solution is not required to satisfy the inner boundary condition, the parameter R_Q is undetermined. There is in fact a one-parameter family of outer ($\delta = 0$) solutions seemingly allowable. They are in a one-to-one correspondence with (i) the interfacial slope at the upstream end, (ii) the downstream thickness or (iii) the fluid-layer volume. For each $R_Q > \frac{3}{2}$, it is possible to construct a boundary-layer correction near $z' = 0$ that matches it. The matching condition determines Q in terms of R_Q. The value $R_Q = 2(1+\eta)$ is characteristic of the $\delta = 0$ problem. This outer solution automatically satisfies the inner boundary condition and so no boundary layer is necessary. The boundary layers for $R_Q \gtrless 2(1+\eta)$ are shown in figures 5 and 6. The boundary-layer structure reflects the importance of the local mean surface tension (in addition to the local gradient) and the thickness of the layer is proportional to this local mean. The mean surface tension enters the problem through the normal force balance across the interface, which within the lubrication approximation is the (dimensional) Laplace relation

$$p = -\sigma/f.$$

This is important in the upstream boundary layer because it produces a non-negligible axial pressure gradient $dp/dz \sim (\sigma/f^2)\, df/dz$, which for $R_Q > 2$ opposes the flow driven by the surface-tension gradient. Hence, in this layer the average downstream velocity is decreased and the layer becomes thicker. When

$$1{\cdot}5 < R_Q < 2{\cdot}0$$

the reverse occurs while for $R_Q < 1{\cdot}5$, no solution could be obtained.

The effect of non-zero surface diffusion is to impose a boundary-layer structure on the downstream end which guarantees a non-zero layer thickness everywhere.

A restriction on the applicability of the analysis arises through the use of the small slope (lubrication) approximation. Figure 6 shows that at the tube ends the slope can, indeed, be large. At the upstream end, $z' = 0$, the slope is small when $R_Q \approx 2(1+\eta)$ since only a gentle boundary layer appears. As R_Q departs further from this value, the quantitative results become less reliable but the qualitative form should still be representative of the solution of the full Navier–Stokes equations within a finite range of R_Q. At the downstream end, $z' = 1$, a vertical slope is found for $\eta = 0$. A sufficiently large amount of surface diffusion would make the slope here gentle as well.

In relation to the fluid flows in alveolar ducts, perhaps the most useful bit of information that could be obtained would be the flow rates. These would be known if R_Q were known. Interestingly, one can identify R_Q in our model simply from a still photograph of the interfacial shape since the R_Q are in a one-to-one correspondence with these. That is, for each interfacial shape, there is a unique value of R_Q.

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